Hi everyone!

I'll be on closer to 1:10, sorry for the delay.

Quadratic Forms :

$q(\vec{x}) = \vec{x}^T A \vec{x}$, where A is a $n \times n$ symmetric matrix

positive definite : $q(\vec{x}) > 0$ (for any $\vec{x} \in \mathbb{R}^n$)

positive semidefinite : $q(\vec{x}) \geq 0$ " "

negative definite : $q(\vec{x}) < 0$ " "

negative semi-def : $q(\vec{x}) \leq 0$ " "

indefinite : $q(\vec{x}) > 0$ & $q(\vec{x}) < 0$ for different vectors $\vec{x}$

Spectral Thm : A, $n \times n$, (real) symmetric $A = A^T$

is (orthogonally) **diagonalizable**

$$A = BDB^{-1}$$

positive definite : $q(\vec{x}) > 0$, all eigenvalues are pos.

positive semidefinite : $q(\vec{x}) \geq 0$, eigenvalues ≥ 0

negative definite : $q(\vec{x}) < 0$, eigenvalues neg

negative semi-def : $q(\vec{x}) \leq 0$, eigenvalues ≤ 0

indefinite : $q(\vec{x}) > 0$ & $q(\vec{x}) < 0$, eigenvalues both pos. & neg.

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$q(\vec{x}) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = \underline{x_1^2} + \underline{4} x_1 x_2 + \underline{x_2^2}$$

$q(\vec{x})$, c_i coeff. on x_i^2, $A = \begin{bmatrix} c_1 & \frac{1}{2}c_{12} & \frac{1}{2}c_{13} & \\ \frac{1}{2}c_{12} & c_2 & & \\ \frac{1}{2}c_{13} & & \ddots & \\ & & & c_n \end{bmatrix}$

c_{ij} coeff. on $x_i x_j$,

i^{th} row, j^{th} col. $(i \neq j)$ is $\frac{1}{2} \cdot$ coeff on $x_i \cdot x_j$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

find eigenvalues

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$\det(A-\lambda I)$ $\rightsquigarrow$

$$(1-\lambda)^2 - 4$$

$$\lambda^2 - 2\lambda + 1 - 4 \rightsquigarrow \lambda^2 - 2\lambda - 3$$

$$(\lambda-3)(\lambda+1)$$

$\lambda = 3, -1$ indefinite

A matrix is diagonalizable if geometric multiplicity of each eigenvalue = algebraic multiplicity of each eigenvalue

g.m : dim. of the corresponding eigenspace

a.m : how many times the eigenvalue appears in the char. poly.

Special cases:

- Matrix is **symmetric**: $A = A^T$, this is always diagonalizable (& orthogonally so)

- If I have an $n \times n$ matrix A, then if A has n distinct eigenvalues, it is diagonalizable.

(nxn matrix $\rightsquigarrow$ deg. n char. poly. $\rightsquigarrow$ has n complex solutions/roots)

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$\underline{\lambda = 3}$: $\ker(A - \lambda I)$

$$= \ker(A - 3I) = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{matrix} \div 2 \\ +(I) \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1 - x_2 = 0 \\ x_2 \text{ free} \end{matrix} \Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = t \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\rightsquigarrow t \in \mathbb{R}$

$$\underline{\overset{E_3 \,\shortparallel}{\ker(A-3I)} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

$$\cancel{\ker(A-3I) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \underline{\lambda = -1}:$$

$$\ker(A + I)$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{matrix} /2 \\ -(I) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 + x_2 = 0 \implies x_1 = -x_2 \\ x_2 \text{ free}, \; x_2 = t \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad E_{-1} = \ker(A + I) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

normalizing eigenspace basis vectors

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{B} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$B^{-1} = B^T$$

Midterm Problem 2:

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y - 3z = 0 \right\}$$

Is W a subspace?

Subspace: A collection of vectors, V, in $\mathbb{R}^n$, that satisfies three conditions:

1. $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in V$

2. V is closed under addition: $\vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V$

3. V is closed under scalar multiplication: $\vec{v} \in V$, $c \in \mathbb{R} \Rightarrow c\vec{v} \in V$.

To check W is a subspace: need to verify <u>all 3</u> conditions for <u>every</u> vector in W

To check W is <u>not</u> a subspace: only need to show one of the conditions fail.

---

① $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y - 3z = 0 \right\}$

$\vec{0} \in W$, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $0 + 2\cdot 0 - 3\cdot 0 = 0$, $0 = 0$ ✓

② $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W \Rightarrow$ $x_1 + 2y_1 - 3z_1 = 0$

$x_2 + 2y_2 - 3z_2 = 0$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in W$$

$$x_1 + x_2 + 2(y_1 + y_2) - 3(z_1 + z_2)$$

$$= x_1 + x_2 + 2y_1 + 2y_2 - 3z_1 - 3z_2 = \underline{x_1 + 2y_1 - 3z_1} \; \underline{+ x_2 + 2y_2 - 3z_2}$$

$$= 0 + 0 = 0$$

3 $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in W$, $x_1 + 2y_1 - 3z_1 = 0$, $c \in \mathbb{R}$

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \qquad cx_1 + 2cy_1 - 3cz_1 = c(x_1 + 2y_1 - 3z_1)$$

$$= c \cdot 0 = 0 \checkmark$$

Basis:

If I have a subspace V, $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ where $\vec{v}_i \in V$ is a **basis** for V if:

① $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$

② $\mathcal{B}$ is a collection of linearly independent vectors

$$\left\{ \begin{array}{l} c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0 \\ \text{These are lin. ind. if the only sol.} \\ \quad \text{is } c_1 = c_2 = \dots = c_n = 0. \end{array} \right\}$$

$\dim(V) = k$, the # of vectors in a basis of V.

ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}$ is this linearly independent?

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

True if the matrix formed by these vectors has no free variables

$\ker(A) = \vec{0}$

Thm: (Rank-Nullity Theorem): A $m \times n$

$$\underbrace{\dim(\operatorname{im}(A))}_{\text{\# of cols. w/ a leading 1}} + \underbrace{\dim(\ker(A))}_{\text{\# of cols. w/ a free variable.}} = n = \text{\# of cols. of } A.$$

ex: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ orthogonality $\Rightarrow$ linear independence

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -(I) \end{array} \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & -1 \end{array}\right]$$

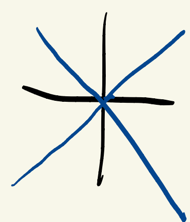
$x_1 - x_2 = 1$

$x_1 + \frac{1}{2} = 1 \Rightarrow x_1 = 1/2$

$2x_2 = -1$

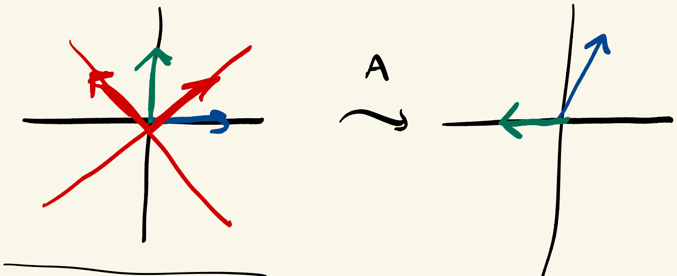
$x_2 = -1/2$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$



$$\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}},\ \underset{\vec{b}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$



$$\boxed{[A]_{\mathcal{B}} = \begin{bmatrix} \underbrace{\left[A \underbrace{b_1}_{\text{change inputs to } \mathcal{B}\text{-basis}}\right]_{\mathcal{B}}}_{\text{change outputs to } \mathcal{B}\text{-basis}} & [Ab_2]_{\mathcal{B}} \end{bmatrix}}$$

$$Ab_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad Ab_2 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 2 \end{array}\right] \begin{array}{c} \\ -(I) \end{array} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 2 \end{array}\right]$$

$x_1 - x_2 = 0$

$x_1 = 1$

$x_2 = 1$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 1 & 1 & -2 \end{array}\right] \begin{array}{c} \\ -(I) \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 2 & 0 \end{array}\right]$$

$x_1 - x_2 = -2$

$x_1 = -2$

$x_2 = 0$

$$\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\boxed{[A]_B = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}}$$

Determinants:

$\det(A) = 0 \iff A$ <u>not</u> invertible

computing this:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

<u>3x3</u>:

$$\begin{array}{cc} \begin{array}{ccc} & + & \end{array} & \begin{array}{cc} - & \end{array} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} & \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \end{array}$$

<u>det</u>:

① swap rows $\rightsquigarrow$ multiplies by (-1)

② scale a <u>row</u> by k $\rightsquigarrow$ scales det. by k

③ Add multiple of one row to another $\rightsquigarrow$ does nothing

8

$$A = BDB^{-1}$$

$$A^n = BD^nB^{-1} = (BD\cancel{B^{-1}})(\cancel{B}DB^{-1}) \cdots (BDB^{-1})$$

$$A^3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix} = B \cdot D^3 \cdot B^{-1}$$

(where $A = BDB^{-1}$)

(5)

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

When does A have <u>two</u> distinct real eigenvalues?

$\det(A - \lambda I)$ ← char. polynomial

A, 2×2, $$\det(A-\lambda I) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) \quad (*)$$

$\mathrm{Tr}(A)$ = sum of the main diagonal.

$$\lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\cdot\det(A)}}{2}$$

$$\mathrm{Tr}(A)^2 > 4\cdot\det(A)$$

$$\Updownarrow$$

$$\mathrm{Tr}(A)^2 - 4\cdot\det(A) > 0$$

$$\Updownarrow$$

$\sqrt{\mathrm{Tr}(A)^2 - 4\cdot\det(A)}$ real.

$$\mathrm{Tr}(A)^2 > 4 \cdot \det(A)$$

$$(a+c)^2 > 4 \cdot (ac - b^2)$$

Ex: $\lambda^2 + 2\lambda + 2$.

$$\lambda^2 + \underbrace{2}_{-\mathrm{Tr}(A)}\lambda + \underbrace{2}_{\det(A)}$$

What is a matrix A whose characteristic polynomial is this?

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

⑥ 1) $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

1 is the only eigenvalue of A.

a.m. of 1 is 3.

$$\overset{\text{dim}}{\downarrow}\ker(A-I) = \underbrace{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}}_{3 \text{ free variables?}}$$

If a matrix A has <u>any</u> non-zero value, then $\dim(\operatorname{im} A) \geq 1$.

$\Rightarrow \quad a = b = c = 0.$

2. $A = \begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$ eigenvalues $= 1, 2, 3$

$\Rightarrow$ 3 distinct e.v. $\Rightarrow$ diagonalizable for all a, b, c.

diagonalizable $\Leftrightarrow$ a.m. = g.m. of each eigenvalue

$1 \leq$ geometric multi $\leq$ alg. multi.